Math2050b HW7

1. On $[-2p, 2p]$, f is continuous and hence uniformly continuous. Hence for $\epsilon > 0$, $\exists\delta > 0$ such that for all $x, y \in [-2p, 2p]$ with $|x - y| < \delta$,

$$|f(x) - f(y)| < \epsilon.$$

Let $x, y \in \mathbb{R}$ with $|x-y| < \delta$. Suppose $x \in [np, (n+1)p)$, then $y \in [(n-1)p, (n+2)p)$ by choosing a smaller $\delta < p/2$. Hence,

$$|f(x) - f(y)| = |f(x - np) - f(y - np)| < \epsilon.$$

2. Let $\epsilon > 0$, there is N such that for all $x, y > N$,

$$|f(x) - f(y)| < \epsilon.$$

On the other hand, for all $x, y \leq a$,

$$|f(x) - f(y)| \leq \kappa |x - y|.$$

Moreover, on $[a - 1, N + 1]$, f is uniformly continuous. That is $\exists 1 > \delta > 0$ such that for all $x, y \in [a - 1, N + 1]$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| < \epsilon.$$

We may assume $\delta < \epsilon/\kappa$. Combine all, we know that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta$. If $x \in [a, N]$, then $y \in [a - 1, N + 1]$, hence $|f(x) - f(y)| < \epsilon$. If $x \leq a$, then either $y \leq a$ or $y \in [a, a + 1]$. In any cases, we still have

$$|f(x) - f(y)| < \epsilon.$$

The last case is when $x > N$, then either $y > N$ or $y \in [N - 1, N]$ where we still have estimate on $|f(x) - f(y)|$.